A Review of Complex Numbers

Complex numbers arose from trying to find roots of polynomials. If we include complex numbers, then any n^{th} degree polynomial has exactly n roots if we count multiplicity. We can see that complex numbers are required when we consider the quadratic polynomial

$$x^2 + 1 = 0$$

Clearly, the solution to this equation gives $x = \sqrt{-1}$ which does not have a value in the real numbers.

To solve this equation, we can define the *imaginary number* $i = \sqrt{-1}$, and create a new number system, called *complex numbers*, which consist of all numbers $z = x + iy$ where z is a complex number, and x and y are real numbers. We say that the *real part* of $z = x + iy$ is x, denoted by $Re(z) = x$, and the *imaginary part* of $z = x + iy$ is y, denoted by $Im(z) = y$. The *conjugate* of a complex number z is denoted by a bar over the number and is computed by replacing every occurance of i with $-i$. Thus, $\overline{x + iy} = x - iy$ and $\overline{e^{\alpha+i\beta}} = e^{\alpha-i\beta}$ for example.

Arithmetic with complex numbers proceeds as if dealing with real numbers. Addition and subtraction remain unchanged, and for multiplication and division we make use of the fact that $i^2 = -1$.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + ix_2y_1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

$$\begin{aligned} z^{-1} &= \frac{1}{x + iy}\left(\frac{x - iy}{x - iy}\right) = \frac{x - iy}{(x + iy)(x - iy)} \\ &= \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} \end{aligned}$$

Note the use of the conjugate to make the denominator real in the third example.

In addition to the representation of a complex number as $z = x + iy$, there is also the *polar representation* $z = Re^{i\theta}$. The polar representation is often easier to use due to its simpler multiplication expression

$$z_1 z_2 = R_1 e^{i\theta_1} R_2 e^{i\theta_2} = (R_1 R_2) e^{i(\theta_1+\theta_2)}$$

The conversion between $z = x + iy$ and $z = Re^{i\theta}$ is done via Euler's formula given by

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Thus, we get the conversions given below:

$$R = \sqrt{x^2 + y^2} \qquad x = R\cos(\theta)$$

$$\theta = \tan^{-1}(y/x) \qquad y = R\sin(\theta)$$